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Knizhnik-Zamolodchikov-type equations for gauged WZNW models

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Abstract

We study correlation functions of coset constructions by utilizing the method of gauge dressing. As an example we apply this method to the minimal models and to the Witten 2D black hole. We exhibit a striking similarity between the latter and the gravitational dressing. In particular, we look for logarithmic operators in the 2D black hole.

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1 Introduction

Gauged Wess-Zumino-Novikov-Witten (WZNW) models [2]-[6] belong to the type of 2D quantum field theories whose correlation functions can be studied systematically in a way which is similar to the Knizhnik-Zamolodchikov (KZ) approach to ordinary WZNW models [1]. This approach being based upon clear physical principles has proved powerful in computation of correlators of WZNW models. Though one has to point out that there are a number of different rather algebraic algorithms one can make use of to find WZNW correlation functions, e.g. [7],[8].

A central element of the KZ method is a differential equation which is now called KZ equation [1]. Correlation functions of the WZNW model turn out to be solutions of this equation. Since gauged WZNW models are closely related to ordinary WZNW models, it is quite natural to conjecture that correlators of the gauged WZNW model have to obey a certain gauge generalization of the KZ equation. In particular, Polyakov [9] has shown that correlation functions of gauged fermions do satisfy a KZ-type equation. At the same time, Witten has proved that free fermions are equivalent to the WZNW model at level one [10]. It is clear that the given equivalence has to hold for the gauged models as well. Hence, correlation functions of the gauged WZNW model at level one has to satisfy the same KZ-type equation of the gauged fermions. In [11], we have derived a differential equation for correlators of the general gauged WZNW model following exactly this intuition. A different generalization of the KZ equation has been obtained in [12]. It is based on the so-called affine-Virasoro construction whose conformal algebra coincides with the one of the stress tensor of the gauged WZNW model (for review see [13]).

Gauged WZNW models have been under extensive consideration for quite a long time mainly because they describe coset constructions [15] which play an important role in string theory and statistical physics. A recent discussion of these models is [16]. While algebraic properties of cosets have been quite well understood, much less is known about their correlation functions (except, perhaps, the case of the minimal models). The coset correlation functions are crucial for understanding of dynamical phenomena in theories such for example as two dimensional black holes [14].

Polyakov's approach to the gauge coupling also has been used to study gravitationally dressed correlation functions of general quantum field theories [17],[18],[20],[21] (see also [19]). Both gravitational and gauge interactions have many features in common. In fact, as we shall show in the present paper, the gravitational dressing can be equivalently described as the gauge dressing. What seems to have been missed so far form the discussion is the mix of gravitational and gauge dressings. This issue will be addressed in the present paper.

In the present paper we would like to present all the details of our derivation of the gauged KZ equation [11] which is the subject of section 2. In section 3, we discuss how the known correlation functions of the minimal models emerge as solutions of our equation. In section 4, we carry on to look for the solutions of our equation in two other interesting cases of parafermions and Witten's 2D black hole. We argue that the latter gives rise to the differential equation which is similar to the equation describing the gravitational dressing. In section 5, we derive an equation which describes the mix of gravitational and gauge dressings. Section 6 contains conclusion and discussion of our results.

2 Gauge dressing of the Knizhnik-Zamolodchikov equation

A large class of 2D conformal field theories is described by gauged WZNW models with the following action (we use here the same normalization as in [16])

$$S(g,A) = S_{WZNW}(g) \; + \; \frac{k}{2\pi}\int d^2z \mathrm{Tr}\left[Ag^{-1}\bar{\partial}g - \bar{A}\partial g g^{-1} + Ag^{-1}\bar{A}g \; - \; A\bar{A}\right], \qquad (2.1)$$

where

$$S_{WZNW}(g) = \frac{k}{8\pi}\int d^2z \; \mathrm{Tr} g^{-1}\partial^{\mu}g g^{-1}\partial_{\mu}g \; + \; \frac{ik}{12\pi}\int d^3z \; \mathrm{Tr} g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg \quad (2.2)$$

and $g\in G$, A, $\bar{A}$ are the gauge fields taking values in the algebra $\mathcal{H}$ of the diagonal group of the direct product $H\times H$, $H\in G$.

It has become usual to study gauged WZNW models with the BRST method [5]. However, this method is not very much of help in computing correlation functions, though,

in principle, the free field realization of these theories allows one to calculate correlators of BRST invariant operators. We shall pursue a different approach which is parallel to the analysis of the gravitational dressing of 2D field theories.

Our starting point are the equations of motion of the gauged WZNW model:

$$\begin{aligned}\bar{\nabla}(\nabla g g^{-1}) &= 0, \\ \bar{\partial}A - \partial\bar{A} + [A,\bar{A}] &= 0,\end{aligned} \tag{2.3}$$

where

$$\bar{\nabla} = \bar{\partial} + \bar{A}, \qquad \nabla = \partial + A. \tag{2.4}$$

Under the gauge symmetry, the WZNW primary fields Φ_i and the gauge fields $A,\ \bar{A}$ transform respectively as follows

$$\begin{aligned}\delta\Phi_i &= \epsilon^a(t_i^a + \bar{t}_i^a)\Phi_i, \\ \delta A &= -\partial\epsilon - [\epsilon, A], \\ \delta\bar{A} &= -\bar{\partial}\epsilon - [\epsilon,\bar{A}],\end{aligned} \tag{2.5}$$

where $t_i^a \in \mathcal{H}$.

In order to fix the gauge invariance, we impose the following condition

$$\bar{A} = 0. \tag{2.6}$$

The given gauge fixing gives rise to the corresponding Faddeev-Popov ghosts with the action

$$S_{ghost} = \int d^2z\ \mathrm{Tr}(b\partial c). \tag{2.7}$$

In the gauge (2.6), the equations of motion take the following form

$$\begin{aligned}\bar{\partial}J &= 0, \\ \bar{\partial}A &= 0,\end{aligned} \tag{2.8}$$

where

$$J = -\frac{k}{2}\partial g g^{-1} - \frac{k}{2} g A g^{-1}. \tag{2.9}$$

Thus, J is a holomorphic current in the gauge (2.6). Moreover, it has canonical commutation relations with the field g and itself:

$$
\begin{aligned}
\{J^a(w), g(z)\} &= t^a g(z)\delta(w,z), \\
\left\{J^a(w), J^b(z)\right\} &= f^{abc}J^c(z)\delta(w,z) \;+\; k/2\delta^{ab}\delta'(w,z).
\end{aligned}
\tag{2.10}
$$

The given commutators follow from the symplectic structure of the gauged WZNW model in the gauge (2.6). In this gauge, the field A plays a role of the parameter v_0 of the orbit of the affine group $\hat{G}$ [22]. Therefore, the symplectic structure of the gauged WZNW model in the gauge (2.6) coincides with the symplectic structure of the original WZNW model [23].

There are residual symmetries which survive the gauge fixing (2.6). Under these symmetries the fields Φ_i and the remaining gauge field A transform according to

$$
\begin{aligned}
\tilde{\delta}\Phi_i &= (\epsilon_L^A t_i^A \;+\; \epsilon_R^a \bar{t}^a)\Phi_i, \\
\tilde{\delta}A &= -\partial\epsilon_R - [\epsilon_R, A],
\end{aligned}
\tag{2.11}
$$

where the parameters ϵ_L and ϵ_R are arbitrary holomorphic functions,

$$
\bar{\partial}\epsilon_{L,R} = 0. \tag{2.12}
$$

In eqs. (2.11) the generators t^A act on the left index of Φ_i, whereas $\bar{t}^a$ act on the right index of Φ_i. One can notice that the left residual group is extended to the whole group G, whereas the right residual group is still the subgroup H.

Eq. (2.9) can be presented in the following form

$$
\frac{1}{2}\partial g \;+\; \frac{\eta}{2}gA \;+\; \frac{1}{\kappa}Jg = 0. \tag{2.13}
$$

Here η and κ are some renormalization constants due to regularization of the singular products gA and Jg.

In order to compute η and κ, we need to do a few things. First of all, we have to find how the gauge field A acts on the fields Φ_i. To this end, let us define dressed correlation

functions

$$\langle\langle\cdots\rangle\rangle \equiv \int \mathcal{D}\bar{A}\mathcal{D}A\langle\cdots\rangle \ \exp\left[-\frac{k}{2\pi}\int d^2z \mathrm{Tr}\left\{\bar{A}g^{-1}\partial g + A\bar{\partial}gg^{-1} + Ag\bar{A}g^{-1} \ + \ A\bar{A}\right\}\right], \tag{2.14}$$

where $\langle\cdots\rangle$ is the correlation function before gauging. The latter is found as a solution to the KZ equation

$$\left\{\frac{1}{2}\frac{\partial}{\partial z_i} \ + \ \sum_{j\neq i}^{N}\frac{t_i^A t_j^A}{k+c_V(G)}\frac{1}{z_i - z_j}\right\}\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,z_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle = 0. \tag{2.15}$$

Here Φ_i are the primary fields of the WZNW model (2.2), t_i^A are the representations of the generators of G for the fields Φ_i,

$$c_V = \frac{f^{abc}f^{abc}}{\dim G}. \tag{2.16}$$

In the gauge (2.6), the dressed correlation functions (2.14) can be presented as follows

$$\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = \int \mathcal{D}b\mathcal{D}c\exp(-S_{ghost}) \ \int \mathcal{D}A\exp[-S_{eff}(A)] \tag{2.17}$$

$$\times\int \mathcal{D}g \ \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N) \ \exp[-\Gamma(g,A)],$$

where $S_{eff}(A)$ is the effective action of the field A and $\Gamma(g,A)$ is formally identical to the original gauged WZNW action in the gauge (2.6). The action S_{eff} is non-local and can be obtained by integration of the following variation (which follows from the Wess-Zumino anomaly condition)

$$\partial\frac{\delta S_{eff}}{\delta A^a} \ + \ f^{abc}A^c\frac{\delta S_{eff}}{\delta A^b} = \tau\bar{\partial}A^a. \tag{2.18}$$

Here the constant τ is to be defined from the consistency condition of the gauge (2.6), which is

$$J_{tot} \equiv \delta Z/\delta\bar{A}^a = 0, \qquad a = 1,2,...,\dim H, \tag{2.19}$$

at $\bar{A} = 0$. Here Z is the partition function of the gauged WZNW model. Condition (2.19) amounts to the vanishing of the central charge of the affine current J_{tot}. This in turn means that J_{tot} is a first class constraint [5]. In order to use this constraint, we need to know the OPE of A with itself. This can be derived as follows. Let us consider the

identity

$$\tau\langle\langle\bar{\partial}A(z)A(z_1)\cdots A(z_N)\rangle\rangle = \int \mathcal{D}A\; A(z_1)\cdots A(z_N)\left[\partial\frac{\delta S_{eff}}{\delta A^a(z)} \;+\; f^{abc}A^c(z)\frac{\delta S_{eff}}{\delta A^b(z)}\right]\mathrm{e}^{-S_{eff}}. \tag{2.20}$$

Here we used relation (2.18). Integrating by parts in the path integral, we arrive at the following formula

$$\begin{aligned}\tau\langle\langle A^a(z)A^{a_1}(z_1)\cdots A^{a_N}(z_N)\rangle\rangle = \frac{1}{2\pi i}\sum_{k=1}^{N}\{&\frac{-\delta^{aa_k}}{(z-z_k)^2}\langle\langle\; A^{a_1}(z_1)\cdots \hat{A}_k^{a_k}\cdots A^{a_N}(z_N)\rangle\rangle \\ &+\frac{f^{aa_kb}}{z-z_k}\langle\langle\; A^{a_1}(z_1)\cdots A_k^{b}\cdots A^{a_N}(z_N)\rangle\rangle\},\end{aligned} \tag{2.21}$$

where $\hat{A}_k$ means that the field $A(z_k)$ is removed from the correlator. In the derivation of the last equation we used the following identity

$$\bar{\partial}_{\bar{z}}\frac{1}{z-z_k} = 2\pi i\delta^{(2)}(z-z_k). \tag{2.22}$$

From eq. (2.21) it follows that

$$\tau A^a(z)A^b(0) = \frac{1}{2\pi i}\left[-\frac{\delta^{ab}}{z^2} \;+\; \frac{f^{abc}}{z}A^c(0)\right] \;+\; \text{reg.} \tag{2.23}$$

Along with condition (2.19), the equation (2.23) gives the expression for τ

$$\tau = \frac{i(k+2c_V(H))}{4\pi}. \tag{2.24}$$

We proceed to derive the Ward identity associated with the residual symmetry (2.11). The Ward identity comes from the change of variables in eq. (2.17) under transformations (2.11). We obtain the following relation

$$\begin{aligned}&\sum_{k=1}^{N}\bar{t}_k^a\delta(z,z_k)\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle \\ &+\tau\langle\langle\bar{\partial}_{\bar{z}}A^a(z)\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = 0.\end{aligned} \tag{2.25}$$

This yields

$$\begin{aligned}&2\pi\tau\langle\langle A^a(z)\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle \\ &= i\sum_{k=1}^{N}\frac{\bar{t}_k^a}{z-z_k}\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle,\end{aligned} \tag{2.26}$$

which in turn gives rise to the OPE between the gauge field A^a and Φ_i

$$\frac{1}{2}A^a(z)\Phi_i(0) = \frac{1}{k+2c_V(H)}\frac{\bar{t}_i^a}{z}\Phi_i(0). \tag{2.27}$$

Now we are in a position to define the product $[A^a, \Phi_i]$. Indeed, we can define this according to the following rule

$$A^a(z)\Phi_i(z,\bar{z}) = \oint \frac{d\zeta}{2\pi i}\frac{A^a(\zeta)\Phi_i(z,\bar{z})}{\zeta - z}, \tag{2.28}$$

where the nominator is understood as OPE (2.27). Formula (2.28) is a definition of normal ordering for the product of two operators.

Let us come back to eq. (2.13). Variation of (2.13) under the residual symmetry with the parameter ϵ_R gives rise to the following relation

$$\left[1 - \eta\left(1 - \frac{c_V(H)}{k+2c_V(H)}\right)\right]\partial\epsilon_R(z)g(z) = 0. \tag{2.29}$$

From this relation we find the renormalization constant η

$$\eta = \frac{k+2c_V(H)}{k+c_V(H)}. \tag{2.30}$$

In the classical limit $k \to \infty$, $\eta \to 1$.

With the given constant η the equation (2.13) reads off

$$\left\{\frac{\partial}{\partial z} \ + \ \frac{k+2c_V(H)}{k+c_V(H)}A(z) \ + \ \frac{2}{\kappa}J(z)\right\}g(z) = 0, \tag{2.31}$$

where $A(z)$ acts on g from the right hand side. In the same fashion, the constant κ can be calculated from variation of (2.13) under the residual symmetry with the parameter ϵ_L, which leads to

$$\kappa = \frac{1}{k+c_V(G)}. \tag{2.32}$$

Note that the given expression for κ is consistent with the condition that the combination $\partial + \eta A$ acted on g as a Virasoro generator L_{-1}:

$$L_{-1}g = \frac{2J^A_{-1}J^A_0}{k+c_V(G)}g, \tag{2.33}$$

where

$$J^A_n = \oint \frac{dz}{2\pi}z^n J^A(z), \tag{2.34}$$

with J^A being defined by eqs. (2.10).

All in all, with the regularization given by eq. (2.28) and the Ward identity (2.26) the eq. (2.31) gives rise to the following differential equation

$$\left\{\frac{1}{2}\frac{\partial}{\partial z_i} + \sum_{j\neq i}^{N}\left(\frac{t_i^A t_j^A}{k+c_V(G)} - \frac{\bar{t}_i^a \bar{t}_j^a}{k+c_V(H)}\right)\frac{1}{z_i - z_j}\right\}\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = 0, \tag{2.35}$$

where $t_i^A \in \mathcal{G}$ and $\bar{t}_i^a \in \mathcal{H}$. An important check of the consistency of this equation is that it should have a flat connection, ie. if we write (2.35) as $\partial_i G = W_i G$, the connection W_i should satisfy

$$\partial_j W_i - \partial_i W_j = [W_i, W_j]. \tag{2.36}$$

Since the connection in W_i in (2.35) is a sum of KZ-type terms, (2.36) follows simply from the fact that $[t^a, \bar{t}^b] = 0$, so the equation is indeed consistent.

Equation (2.35) is our main result. By solving it, one can find dressed correlation functions in the gauged WZNW model. The solutions can be expressed as products of the correlation functions in the WZNW model for the group G at level k and H at level $-2c_V(H) - k$. In particular, for the two-point function the equation yields

$$\frac{1}{2}\partial\langle\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\rangle = -\left[\frac{t_i^A t_j^A}{k+c_V(G)} - \frac{\bar{t}_i^a \bar{t}_j^a}{k+c_V(H)}\right]\frac{1}{z}\langle\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\rangle. \tag{2.37}$$

By the projective symmetry, the two-point function has the following expression

$$\langle\langle\Phi_i(z,\bar{z})\Phi_j(0)\rangle\rangle = \frac{G_{ij}}{|z|^{4\Delta_i}}, \tag{2.38}$$

where Δ_i is the anomalous conformal dimension of Φ_i after the gauge dressing and G_{ij} is the Zamolodchikov metric which can be diagonalized. After substitution of expression (2.38) into eq. (2.37), and using the fact that, as a consequence of the residual symmetry (2.11), the dressed correlation functions must be singlets of both the left residual group G and the right residual group H, we find

$$\Delta_i = \frac{c_i(G)}{k+c_V(G)} - \frac{c_i(H)}{k+c_V(H)}, \tag{2.39}$$

where $c_i(G) = t_i^A t_i^A$, $c_i(H) = \bar{t}_i^a \bar{t}_i^a$.

Up to now we have only considered simple groups, but the above analysis can easily be extended to include semi-simple groups. Of particular interest are G/H coset models

where $G = H \times H$ and the diagonal H subgroup is gauged. In this case the action (2.1) becomes

$$\begin{aligned} S(g,\tilde{g},A) &= S_{WZNW}(g,k_1) \;+\; S_{WZNW}(\tilde{g},k_2) \\ &+ \frac{k_1}{2\pi}\int d^2z \mathrm{Tr}\left[Ag^{-1}\bar{\partial}g - \bar{A}\partial g g^{-1} + Ag^{-1}\bar{A}g \;-\; A\bar{A}\right] \\ &+ \frac{k_2}{2\pi}\int d^2z \mathrm{Tr}\left[A\tilde{g}^{-1}\bar{\partial}\tilde{g} - \bar{A}\partial\tilde{g}\tilde{g}^{-1} + A\tilde{g}^{-1}\bar{A}\tilde{g} \;-\; A\bar{A}\right], \end{aligned} \tag{2.40}$$

where $S_{WZNW}(g,k)$ is the same as in (2.2), $g \in H_{k_1}$ and $\tilde{g} \in H_{k_2}$. The equations of motion in the gauge (2.6) are now

$$\bar{\partial}J_1 = \bar{\partial}J_2 = \bar{\partial}A = 0, \tag{2.41}$$

where,

$$\begin{aligned} J_1 &= -\frac{k_1}{2}\partial g g^{-1} \;-\; \frac{k_1}{2}gAg^{-1} \\ J_2 &= -\frac{k_2}{2}\partial\tilde{g}\tilde{g}^{-1} \;-\; \frac{k_2}{2}\tilde{g}A\tilde{g}^{-1} \end{aligned} \tag{2.42}$$

The residual symmetries (2.11) take the form

$$\begin{aligned} \tilde{\delta}\Phi_i &= (\epsilon_L^a t_i^a \;+\; \tilde{\epsilon}_L^a s_i^a \;+\; \epsilon_R^a\{\bar{t}^a + \bar{s}^a\})\Phi_i, \\ \tilde{\delta}A &= -\partial\epsilon_R - [\epsilon_R, A], \end{aligned} \tag{2.43}$$

where t^a are the generators for H_{k_1} and s^a are the generators for H_{k_2}, so that $t^a + s^a$ are the generators of the diagonal subgroup.

As in eq. (2.13), eq. (2.42) can be presented in the form

$$\frac{1}{2}\partial(g\tilde{g}) \;+\; \frac{\eta}{2}(g\tilde{g})A \;+\; \frac{1}{\kappa_1}J_1 g\tilde{g} \;+\; \frac{1}{\kappa_2}J_2 g\tilde{g} = 0. \tag{2.44}$$

The renormalization constants η, κ_1 and κ_2 can be found in the same way as for simple groups, giving a differential equation similar to (2.35).

$$\begin{aligned} &\left\{\frac{1}{2}\frac{\partial}{\partial z_i} \;+\; \sum_{j\neq i}^{N}\left(\frac{t_i^a t_j^a}{k_1 + c_V(H)} + \frac{s_i^a s_j^a}{k_2 + c_V(H)} - \frac{(\bar{t}_i^a + \bar{s}_i^a)(\bar{t}_j^a + \bar{s}_j^a)}{k_1 + k_2 + c_V(H)}\right)\frac{1}{z_i - z_j}\right\} \times \\ &\qquad\qquad \times\langle\langle\Phi_1(z_1,\bar{z}_1)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = 0. \end{aligned} \tag{2.45}$$

Note that we have derived an equation (2.35) or (2.45) for the holomorphic part of the correlation function only. This is because we started from the gauge (2.6). We could also have used the gauge $A = 0$, in which case we would have derived the same equation with z replaced by $\bar{z}$, and t^a by $\bar{t}^a$. Therefore for gauge invariant correlation functions the holomorphic and antiholomorphic parts will be a solutions to (2.35).

3 Minimal Models

To illustrate the use of (and as a check of the correctness of) equation (2.45), we consider the unitary minimal series of models, with central charge $c = 1 - \frac{6}{m(m+1)}$, which is well known to coincide with the coset model for $\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}$, with $m = k+2$. The dimensions of primary fields in these models are given by the Kac formula, which can be presented in the form

$$h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)} = \frac{j(j+1)}{k+2} - \frac{j'(j'+1)}{k+3} + \frac{\epsilon(\epsilon+1)}{3} + n^2 \tag{3.46}$$

Here $r = 2j+1$, $s = 2j'+1$, $\epsilon = 0$ and $n^2 = (j-j')^2$ if $j - j'$ is an integer, and $\epsilon = 1/2$ and $n^2 = (j-j')^2 - \frac{1}{4}$ if $j - j'$ is a half integer. r, s are integers with $1 \leq r \leq m$ and $1 \leq s \leq m+1$.

We shall construct fields in the minimal model from primary fields of the WZNW model as follows. We observe that the primary field ϕ_j with isospin j under $SU(2)_k$ and a singlet under $SU(2)_1$, will have isospin j in $SU(2)_{k+1}$ and so, according to eq.(2.39) will have dimension

$$h_j = \frac{j(j+1)}{k+2} - \frac{j(j+1)}{k+3} = h_{r,r}. \tag{3.47}$$

The primary fields $\phi_j \times \tilde{g}$, with isospin j under $SU(2)_k$ and isospin $1/2$ under $SU(2)_1$, have two components, with isospin $j \pm 1/2$ in $SU(2)_{k+1}$, giving the dimensions

$$h_{j\pm} = \frac{j(j+1)}{k+2} - \frac{(j \pm \frac{1}{2})((j \pm \frac{1}{2}) + 1)}{k+3} + \frac{1}{4} = h_{r,r\pm 1} \tag{3.48}$$

We can therefore identify the fields $\Phi_{r,r}$ and $\Phi_{r,r\pm 1}$ as

$$\begin{aligned} \Phi_{r,r} &= Tr\phi_j \\ \Phi_{r,r\pm 1} &= Tr\{\phi_j \times \tilde{g}\}_{j\pm\frac{1}{2}} \end{aligned} \tag{3.49}$$

In the WZNW model for $SU(2)_k$ the primary fields have isospins $0, \frac{1}{2}, \ldots, \frac{k}{2}$, giving $1 \leq r \leq m$ and $1 \leq s \leq m+1$ as expected. For $SU(2)_1$ there is only $j = 0$ or $\frac{1}{2}$, so we cannot construct any other fields in this way. The differential equation (2.45) for a correlation function of these fields is

$$\left\{ \frac{1}{2}\frac{\partial}{\partial z_i} + \sum_{j \neq i}^{N} \left(\frac{t_i^A t_j^A}{k+2} + \frac{s_i^A s_j^A}{3} - \frac{(\bar{t}_i^A + \bar{s}_i^A)(\bar{t}_j^A + \bar{s}_j^A)}{k+3} \right) \frac{1}{z_i - z_j} \right\} \times \langle\langle \Phi_1(z_1, \bar{z}_1) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle = 0, \tag{3.50}$$

t^A and s^A are the generators of the $SU(2)_k$ and $SU(2)_1$ respectively. All the solutions to eq. (3.50) can be written as products of correlation functions of primary fields in the WZNW model at levels k, 1, and $(-k-5)$ (the $(-k-5)$ factor comes from writing $-\frac{1}{k+3} = \frac{1}{(-k-5)+2}$);

$$\langle\langle \Phi_1(z_1, \bar{z}_1) \cdots \Phi_N(z_N, \bar{z}_N) \rangle\rangle = f_k(z_1, \ldots, z_N) f_1(z_1, \ldots, z_N) f_{-k-5}(z_1, \ldots, z_N) \tag{3.51}$$

where $f_k(z_1, \ldots, z_N)$ is a solution to the KZ equation for $SU(2)_k$:

$$\begin{aligned} \left\{ \frac{1}{2}\frac{\partial}{\partial z_i} + \frac{1}{z_i - z_j} \sum_{j \neq i}^{N} \frac{t_i^A t_j^A}{k+2} \right\} f_k(z_1, \ldots, z_N) &= 0 \\ \left\{ \frac{1}{2}\frac{\partial}{\partial z_i} + \frac{1}{z_i - z_j} \sum_{j \neq i}^{N} \frac{s_i^A s_j^A}{3} \right\} f_1(z_1, \ldots, z_N) &= 0 \\ \left\{ \frac{1}{2}\frac{\partial}{\partial z_i} - \frac{1}{z_i - z_j} \sum_{j \neq i}^{N} \frac{(\bar{t}_i^A + \bar{s}_i^A)(\bar{t}_j^A + \bar{s}_j^A)}{k+3} \right\} f_{-k-5}(z_1, \ldots, z_N) &= 0 \end{aligned} \tag{3.52}$$

It was already shown in [2] that the coset model factorises into $SU(2)_k$, $SU(2)_1$ and $SU(2)_{-k-5}$ sectors.

The simplest correlation functions to calculate in this way are the functions for the fields $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$. The 4-point functions for these fields can be built up from the conformal blocks in the $SU(2)$ WZNW model for the field in the fundamental representation of $SU(2)$. These are [1]:

$$\langle g_{\epsilon_1\bar{\epsilon}_1}(z_1, \bar{z}_1) g^\dagger_{\bar{\epsilon}_2\epsilon_2}(z_2, \bar{z}_2) g_{\epsilon_3\bar{\epsilon}_3}(z_3, \bar{z}_3) g^\dagger_{\bar{\epsilon}_4\epsilon_4}(z_4, \bar{z}_4) \rangle = \frac{1}{|z_{14}z_{23}|^{4\Delta}} \sum_{A,B=0,1} G_{AB}(x, \bar{x}) I_A \bar{I}_B$$

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \qquad \Delta = \frac{3}{4(2+k)} \qquad I_1 = \delta_{\epsilon_1\epsilon_2}\delta_{\epsilon_3\epsilon_4} \qquad I_2 = \delta_{\epsilon_1\epsilon_4}\delta_{\epsilon_2\epsilon_3}$$

$$
\begin{aligned}
G_{AB}(x,\bar{x}) &= \sum_{p,q=0,1} U_{pq} F^{(p)}_{A,[k]}(x) F^{(q)}_{B,[k]}(\bar{x}) \\
F^{(0)}_{1,[k]}(x) &= x^{-\frac{3}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}} F(\frac{1}{2+k}, -\frac{1}{2+k}; \frac{k}{2+k}; x) \\
F^{(0)}_{2,[k]}(x) &= \frac{1}{k} x^{\frac{1+2k}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}} F(\frac{1+k}{2+k}, \frac{3+k}{2+k}; \frac{2+2k}{2+k}; x) \\
F^{(1)}_{1,[k]}(x) &= x^{\frac{1}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}} F(\frac{1}{2+k}, \frac{3}{2+k}; \frac{4+k}{2+k}; x) \\
F^{(1)}_{2,[k]}(x) &= -2x^{\frac{1}{2(2+k)}}(1-x)^{\frac{1}{2(2+k)}} F(\frac{1}{2+k}, \frac{3}{2+k}; \frac{2}{2+k}; x) \\
U_{10} &= U_{01} = 0 \quad U_{11} = hU_{00} \\
h &= \frac{1}{4} \frac{\Gamma(\frac{1}{2+k})\Gamma(\frac{3}{2+k})}{\Gamma(\frac{1+k}{2+k})\Gamma(\frac{-1+k}{2+k})} \frac{\Gamma^2(\frac{k}{2+k})}{\Gamma^2(\frac{2}{2+k})}
\end{aligned} \tag{3.53}
$$

In the case of $k=1$, The conformal blocks $F^{(1)}_{A,[1]}(x)$ are excluded, and the remaining functions reduce to

$$
\begin{aligned}
F_{1,[1]}(x) &= \left(\frac{1-x}{x}\right)^{\frac{1}{2}} \\
F_{2,[1]}(x) &= \left(\frac{x}{1-x}\right)^{\frac{1}{2}}
\end{aligned} \tag{3.54}
$$

The first example we consider is the four-point function of the $\Phi_{1,2}$ field. By equation (3.49), $\Phi_{1,2} = \tilde{g}_{\epsilon\bar{\epsilon}}\delta_{\epsilon\bar{\epsilon}}$, and so we find from equations (3.50) and (3.53) the following conformal blocks:

$$
\begin{aligned}
(z_{14}z_{23})^{-2h_{1,2}} \mathcal{F}^{(p)}_{(1,2)}(x) &\sim \langle\langle \Phi_{1,2}(z_1)\Phi_{1,2}(z_2)\Phi_{1,2}(z_3)\Phi_{1,2}(z_4)\rangle\rangle \\
\mathcal{F}^{(p)}_{(1,2)}(x) &= 2F^{(p)}_{1,[-k-5]}(x)F_{1,[1]}(x) + 2F^{(p)}_{2,[-k-5]}F_{2,[1]}(x) + \\
&\quad + F^{(p)}_{1,[-k-5]}(x)F_{2,[1]}(x) + F^{(p)}_{2,[-k-5]}(x)F_{1,[1]}(x)
\end{aligned} \tag{3.55}
$$

The expression for the conformal blocks for the fields $\Phi_{2,1} = g_{\epsilon_1\bar{\epsilon}_1}\tilde{g}_{\epsilon_2\bar{\epsilon}_2}\delta_{\epsilon_1\epsilon_2}\delta_{\bar{\epsilon}_1\bar{\epsilon}_2}$ and $\Phi_{2,2} = g_{\epsilon\bar{\epsilon}}\delta_{\epsilon\bar{\epsilon}}$, are similar:

$$
\begin{aligned}
(z_{14}z_{23})^{-2h_{2,1}} \mathcal{F}^{(p)}_{(2,1)}(x) &\sim \langle\langle \Phi_{2,1}(z_1)\Phi_{2,1}(z_2)\Phi_{2,1}(z_3)\Phi_{2,1}(z_4)\rangle\rangle \\
\mathcal{F}^{(p)}_{(2,1)}(x) &= 2F^{(p)}_{1,[k]}(x)F_{1,[1]}(x) + 2F^{(p)}_{2,[k]}F_{2,[1]}(x) + \\
&\quad + F^{(p)}_{1,[k]}(x)F_{2,[1]}(x) + F^{(p)}_{2,[k]}(x)F_{1,[1]}(x) \\
(z_{14}z_{23})^{-2h_{2,2}} \mathcal{F}^{(p,q)}_{(2,2)}(x) &\sim \langle\langle \Phi_{2,2}(z_1)\Phi_{2,2}(z_2)\Phi_{2,2}(z_3)\Phi_{2,2}(z_4)\rangle\rangle \\
\mathcal{F}^{(p,q)}_{(2,2)}(x) &= 2F^{(p)}_{1,[k]}(x)F^{(q)}_{1,[-k-5]}(x) + 2F^{(p)}_{2,[k]}F^{(q)}_{2,[-k-5]}(x) + \\
&\quad + F^{(p)}_{1,[k]}(x)F^{(q)}_{2,[-k-5]}(x) + F^{(p)}_{2,[k]}(x)F^{(q)}_{1,[-k-5]}(x)
\end{aligned} \tag{3.56}
$$

This gives two solutions for the conformal blocks for the $\Phi_{1,2}$ and $\Phi_{2,1}$ fields, and four for the $\Phi_{2,2}$ field, but the equation (3.50) is actually a fourth order differential equation in all three cases, and so there are two more solutions (which are given by replacing $F_{A,[1]}(x)$ by $F^{(1)}_{A,[1]}(x)$ in (3.55) and (3.56)). When the $\bar{x}$ dependence is restored by imposing conditions of crossing symmetry and locality, the two extra solutions do not contribute to the full 4-point function. Using eq. (3.54), the conformal blocks for the $\Phi_{1,2}$ and $\Phi_{2,1}$ fields can be reduced to expressions containing only one hypergeometric function, which agree with the conformal blocks given in [27].

$$\begin{aligned}
\mathcal{F}^{(0)}_{(1,2)}(x) &= 2x^{-2h_{1,2}}(1-x)^{h_{1,3}-2h_{1,2}}F\left(\frac{k+2}{k+3},\frac{1}{k+3};\frac{2}{k+3};x\right)\\
\mathcal{F}^{(1)}_{(1,2)}(x) &= \frac{3k}{k+1}[x(1-x)]^{h_{1,3}-2h_{1,2}}F\left(\frac{k+2}{k+3},\frac{2k+3}{k+3};\frac{2k+4}{k+3};x\right)\\
\mathcal{F}^{(0)}_{(2,1)}(x) &= 2x^{-2h_{2,1}}(1-x)^{h_{3,1}-2h_{2,1}}F\left(\frac{k+3}{k+2},\frac{-1}{k+2};\frac{-2}{k+2};x\right)\\
\mathcal{F}^{(1)}_{(2,1)}(x) &= \frac{3(k+5)}{k+4}[x(1-x)]^{h_{3,1}-2h_{2,1}}F\left(\frac{k+3}{k+2},\frac{2k+7}{k+2};\frac{2k+6}{k+2};x\right)
\end{aligned} \tag{3.57}$$

The trace in eq. (3.49) ensures that all fields are gauge singlets. If instead we simply considered $g_{\epsilon,\bar{\epsilon}}$ or $\tilde{g}_{\epsilon,\bar{\epsilon}}$, we would find similar solutions to (3.50) but there are no simultaneous non-trivial solutions to the equation for the $\bar{x}$ dependence. In this way we can express all correlation functions of $\Phi_{r,r}$ fields in terms of correlation functions in $SU(2)_k$ and $SU(2)_{-k-5}$ WZNW models, and functions of $\Phi_{r,r\pm1}$ in terms of functions in $SU(2)_k$, $SU(2)_1$ and $SU(2)_{-k-5}$ WZNW models. Equation (3.50) does not apply to fields with $|r-s|>1$, since in deriving eq. (2.45) we assumed that we were dealing with primary fields of the ungauged WZNW model. However, using the relation $h_{r,s}=h_{m-r,m+1-s}$, fields with $s=r+2$ can also be considered as fields with $r-s=1$, ie. $\Phi_{r,r+2}=\Phi_{m-r,m-r-1}$, and so we can also use (3.50) to find correlation functions involving these fields. It can be seen from eq. (3.46) that the fields which obey eq. (3.50) include all the relevant fields ($h_{r,s}<1$) in the unitary minimal models. Although we cannot compute all correlation functions directly from (3.50), that equation nevertheless does in principle contain complete information about all correlation functions, as all the fields $\phi_{r,s}$ can be obtained from the operator product expansion of fields with $|r-s|\leq 1$. For example, using the

OPE

$$\Phi_{2,1} \times \Phi_{2,1} \sim [I] + [\Phi_{3,1}], \tag{3.58}$$

which can be deduced from (3.57), we can obtain the n-point function of $\Phi_{3,1}$ fields from the 2n-point function of $\Phi_{2,1}$ fields.

4 $SU(2)/U(1)$ and $SL(2,\mathcal{R})/U(1)$ Models

In this section we discuss the use of eq.(2.35) in the closely related coset models for $SU(2)/U(1)$ and $SL(2,\mathcal{R})/U(1)$. In both cases the central charge is

$$c = \frac{3k}{k+2} - 1, \tag{4.59}$$

but in the case of $SU(2)/U(1)$ k is a positive integer, while in $SL(2,\mathcal{R})/U(1)$ k is negative and not necessarily an integer. In particular, $k = -9/4$, $c = 26$ gives the 2D back hole [14]. The $SU(2)/U(1)$ model with integer k is the $\mathcal{Z}_k$ parafermion model of [28]. Our convention for the sign of k in $SL(2,\mathcal{R})/U(1)$ is opposite to [14], so that we can use the same equations for both $SU(2)/U(1)$ and $SL(2,\mathcal{R})/U(1)$. Also, in $SL(2,\mathcal{R})/U(1)$ the $U(1)$ subgroup can be either compact or non-compact. In the case when $U(1)$ is compact, our equation takes the following form

$$\left\{\frac{\partial}{\partial z_i} \; + \; 2\sum_{j\neq i}^{N}\left(\frac{t_i^A t_j^A}{k+2} - \frac{\bar{t}_i^3 \bar{t}_j^3}{k}\right)\frac{1}{z_i - z_j}\right\}\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = 0, \tag{4.60}$$

where $t_i^A \in \mathcal{SL}(2)$. While in the non-compact case, the equation is

$$\left\{\frac{\partial}{\partial z_i} \; + \; 2\sum_{j\neq i}^{N}\left(\frac{t_i^A t_j^A}{k+2} + \frac{\bar{t}_i^3 \bar{t}_j^3}{k}\right)\frac{1}{z_i - z_j}\right\}\langle\langle\Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2)\cdots\Phi_N(z_N,\bar{z}_N)\rangle\rangle = 0, \tag{4.61}$$

in the case of $SU(2)/U(1)$, the $U(1)$ subgroup is always compact, and so the equation is (4.60) with $t_i^A \in \mathcal{SU}(2)$. The conformal dimension given by eq. (2.39) for a field $\Phi_{j,m}^{j,\bar{m}}$, with $SU(2)$ or $SL(2,\mathcal{R})$ isospin j and $t^3\Phi_{j,m}^{j,\bar{m}} = m\Phi_{j,m}^{j,\bar{m}}$, $\bar{t}^3\Phi_{j,m}^{j,\bar{m}} = \bar{m}\Phi_{j,m}^{j,\bar{m}}$ is

$$\Delta_j^{\bar{m}} = \frac{j(j+1)}{k+2} - g_{33}\frac{\bar{m}^2}{k}. \tag{4.62}$$

In the case of a compact $U(1)$, $g_{33} = +1$, and in the non-compact case $g_{33} = -1$. In the case of $SU(2)$, j is of course an integer or half-integer with $0 \leq j \leq k/2$, and, $m = -j, -j+1, \ldots, j$. However, for the $SL(2,\mathcal{R})/U(1)$ model of $2D$ black holes [14] we are most interested in the infinite dimensional representations of $SL(2,\mathcal{R})$, and so j can take any value. If the $U(1)$ group is non-compact, correlation functions will only be gauge invariant, (and satisfy the differential equations for both z and $\bar{z}$) if $m + \bar{m} = 0$, while if the $U(1)$ group is compact, we can have $m + \bar{m} = nk$ for integer n. Of course, in the case of $SU(2)/U(1)$ $|m| \leq k/2$ anyway, so we can only have $n = 0$.

As with the minimal models, the solutions to eqs. (4.60) and (4.61) can be expressed as products of conformal blocks from $SU(2)_k$ or $SL(2,\mathcal{R})_k$ and $U(1)_{-k}$ WZNW models, giving:

$$\langle\langle \Phi_{j_1,m_1}^{j_1,\bar{m}_1}(z_1) \ldots \Phi_{j_N,m_N}^{j_N,\bar{m}_N}(z_N) \rangle\rangle \sim \prod_{i<j} (z_i - z_j)^{\frac{2g_{33}\bar{m}_i\bar{m}_j}{k}} G_{m_1,\ldots,m_N}(z_1, \ldots, z_N). \tag{4.63}$$

Where $G_{m_1,\ldots,m_N}(z_1, \ldots, z_N)$ is a solution to the KZ equation for $SU(2)_k$ or $SL(2,\mathcal{R})_k$, and the prefactor is a $U(1)_{-k}$ correlation function. The simplest example is the four-point function for the field in the fundamental representation of $SU(2)$, $\Phi = \Phi_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2},-\frac{1}{2}} + \Phi_{\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2},\frac{1}{2}}$. In order to use (4.63) it is convenient to write the $SU(2)_k$ conformal blocks (3.53) as

$$\begin{aligned}
F_{+-+-}^{(a)}(x) &= F_{1,[k]}^{(a)}(x) \\
F_{++--}^{(a)}(x) &= F_{2,[k]}^{(a)}(x) \\
F_{+--+}^{(a)}(x) &= -F_{1,[k]}^{(a)}(x) - F_{2,[k]}^{(a)}(x).
\end{aligned} \tag{4.64}$$

Using eqs. (3.53), (4.63) and (4.64), the conformal blocks in the $SU(2)/U(1)$ model can now be written as :

$$\begin{aligned}
\mathcal{F}^{(0,1)}(x) &= x^{-2\Delta}(1-x)^{\Delta_1 - 2\Delta} F\left(\frac{1}{2+k}, -\frac{1}{2+k}; \frac{k}{2+k}; x\right) \\
\mathcal{F}^{(0,2)}(x) &= \frac{1}{k} x^{\Delta_2 - 2\Delta}(1-x)^{\Delta_3 - 2\Delta} F\left(\frac{1+k}{2+k}, \frac{3+k}{2+k}; \frac{2+2k}{2+k}; x\right) \\
\mathcal{F}^{(1,1)}(x) &= x^{\Delta_3 - 2\Delta}(1-x)^{\Delta_1 - 2\Delta} F\left(\frac{1}{2+k}, \frac{3}{2+k}; \frac{4+k}{2+k}; x\right) \\
\mathcal{F}^{(1,2)}(x) &= -2x^{\Delta_1 - 2\Delta}(1-x)^{\Delta_3 - 2\Delta} F\left(\frac{1}{2+k}, \frac{3}{2+k}; \frac{2}{2+k}; x\right)
\end{aligned} \tag{4.65}$$

where $\Delta_1 = \frac{2}{k+2} - \frac{1}{k} = \Delta_1^1$, $\Delta_2 = 1 - \frac{1}{k} = \Delta_{k/2}^{k/2-1}$, and $\Delta_3 = \frac{2}{k+2} = \Delta_1^0$. When $k = 1$, $c = 0$ and $\Delta = 0$, and (4.65) simplifies to $\mathcal{F}^{(0,1)}(x) = \mathcal{F}^{(0,2)}(x) = 1$. When $k = 2$, $c = \frac{1}{2}$, (4.65) reduces to the conformal blocks for the $\Phi_{1,2}$ field for the minimal model with $k = 1$ (3.57). The full four-point function must obey both eq. (4.60) and, because of gauge invariance, the same equation with z_i, t_i^A and $\bar{t}_i^3$ replaced by $\bar{z}_i$, $\bar{t}_i^A$ and t_i^3 respectively. The general solution is:

$$\langle\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle\rangle = (z_{14}z_{23})^{-2\Delta} \sum_{a,b=0}^{1} \left[U_{a,b}^1 \mathcal{F}^{(a,1)}(x)\mathcal{F}^{(b,1)}(\bar{x}) + U_{a,b}^2 \mathcal{F}^{(a,2)}(x)\mathcal{F}^{(b,2)}(\bar{x}) \right]. \tag{4.66}$$

The constraints of locality at $x = 0, 1$ and crossing symmetry imply

$$U_{0,1}^A =_{1,0}^A = 0, \qquad U_{1,1}^A = hU_{0,0}^A, \qquad U_{0,0}^1 = U_{0,0}^2, \qquad A = 1, 2 \tag{4.67}$$

h is the same as in eq. (3.53). Eq. (4.66) does not contain terms such as $\mathcal{F}^{(a,1)}(x)\mathcal{F}^{(a,2)}(\bar{x})$, even before locality and crossing symmetry are imposed, because this would be a solution to eq. (4.60) for the function $\langle\langle \Phi_+^-(1)\Phi_-^+(2)\Phi_+^-(3)\Phi_-^+(4)\rangle\rangle$, and to the equation for $\bar{x}$ for the different function $\langle\langle \Phi_+^-(1)\Phi_+^-(2)\Phi_-^+(3)\Phi_-^+(4)\rangle\rangle$.

The solution above also applies in the case of $SL(2,\mathcal{R})/U(1)$ with a compact $U(1)$, but correlation functions of fields in finite dimensional representations of $SL(2,\mathcal{R})$ are not very useful for the study of the 2D black hole. In order to study the infinite dimensional representations of the coset $SL(2)/U(1)$, we make use of the following representation of the $SL(2)$ generators:

$$\begin{aligned} t^+ &= \frac{\partial}{\partial y}, \\ t^- &= y^2\frac{\partial}{\partial y} - 2jy, \\ t^3 &= y\frac{\partial}{\partial y} - j. \end{aligned} \tag{4.68}$$

Then, the residual symmetries impose the following constraints

$$\sum_{n=1}^{N} \left[y_n^{l+1}\frac{\partial}{\partial y_n} - (l+1)jy_n^l \right] \langle\langle \Phi_1(z_1, y_1)...\Phi_2(z_2, y_2)\rangle\rangle = 0, \tag{4.69}$$

where $l = -1,\ 0,\ 1$. These constraints are nothing but the Ward identities of the $SL(2)$ projective conformal group acting on parameters y_n. In particular, the four-point function

depending on $y_1,\ y_2,\ y_3,\ y_4$, due to the Ward identities, can be presented as follows

$$G(y_1,\ y_2,\ y_3,\ y_4) = \left[(y_1 - y_4)(y_2 - y_3)\right]^{2j} F(t), \tag{4.70}$$

where

$$t = \frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)}, \tag{4.71}$$

the function F depends only on t , whereas j is the $SL(2)$ spin of the operator Φ. With the representation (4.68), eq. (4.60) takes the form

$$\left\{ \frac{1}{2}\frac{\partial}{\partial z_i} + \sum_{j\neq i}^{N} \frac{1}{z_i - z_j}\left[\frac{1}{2(k+2)}\left(-(y_i - y_j)^2\frac{\partial^2}{\partial y_i \partial y_j} - 2(y_i - y_j)\left(j_j\frac{\partial}{\partial y_i} - j_i\frac{\partial}{\partial y_j}\right) + 2j_i j_j\right)\right.\right.$$
$$\left.\left. -\frac{1}{k}\left(\bar{y}_i\bar{y}_j\frac{\partial^2}{\partial \bar{y}_i\partial \bar{y}_j} - j_i\bar{y}_j\frac{\partial}{\partial \bar{y}_j} - j_j\bar{y}_i\frac{\partial}{\partial \bar{y}_i} + j_i j_j\right)\right]\right\}\langle\langle \Phi_1(1)\cdots\Phi_N(N)\rangle\rangle = 0. \tag{4.72}$$

Here j_i is the $SL(2)$ spin of the operator Φ_i, $\Phi(1) = \Phi(z_1, \bar{z}_1, y_1, \bar{y}_1)$ and we have used the representation (4.68) for both left and right $SL(2)$ groups, so $\bar{t}^3$ becomes $\bar{y}\frac{\partial}{\partial \bar{y}} - j$. As long as we are only interested in the z and not the $\bar{z}$ dependence of the correlation functions, we can replace $\bar{t}^3_i$ by the $U(1)$ charge $\bar{m}_i$. The solutions to eq. (4.72) can then, as before, be written as products of $SL(2)$ and $U(1)$ factors:

$$\langle\langle \Phi_1^{\bar{m}_1}(1)\cdots\Phi_N^{\bar{m}_N}(N)\rangle\rangle \sim F_{SL(2)}(z_1, y_1, \ldots, z_N, y_N)\prod_{i<j}(z_i - z_j)^{\frac{2g_{33}\bar{m}_i\bar{m}_j}{k}} \tag{4.73}$$

Where $F_{SL(2)}(z_1, y_1, \ldots, z_N, y_N)$ satisfies

$$\left\{(k+2)\frac{\partial}{\partial z_i} + \sum_{j\neq i}^{N}\frac{1}{z_i - z_j}\left(-(y_i - y_j)^2\frac{\partial^2}{\partial y_i\partial y_j} - 2(y_i - y_j)\left(j_j\frac{\partial}{\partial y_i} - j_i\frac{\partial}{\partial y_j}\right) + 2j_i j_j\right)\right\}F_{SL(2)} = 0 \tag{4.74}$$

Equation (4.74) is closely related to the equation for correlation functions in gravitationally dressed CFTs derived in [21]. Indeed, if we put $j_i = -\Delta$, $k + 2 = -\gamma$, $z_i = x_i^+$ and $y_i = z_i^-$ in (4.74), we get equation (2.41) from [21]*. The important difference is that in the case of gravitationally dressed CFTs both the coordinates appearing in (4.74) are world-sheet coordinates, while in the present case y is an extra $SL(2)$ coordinate. Another unusual feature of eq.(4.72) is that the fields Φ_i each contain a number of different

*In eqs (4.72) and (4.74) we have used $t^A t^A \equiv \eta_{\alpha\beta}t^\alpha t^\beta$, with $\eta_{33} = 1$, $\eta_{+-} = \eta_{-+} = -1/2$. Since k is negative, this gives the $SL(2,\mathcal{R})$ metric $k\eta_{\alpha\beta}$ the signature $(++-)$.

primary fields (because Φ_i has components with different $U(1)$ charges. To see this, we write Φ_i as

$$\Phi(z,\bar{z},y,\bar{y}) = \sum_{m,\bar{m}} \Phi^{\bar{m}}_{m}(z,\bar{z}) y^{j+m} \bar{y}^{j+\bar{m}}. \tag{4.75}$$

The solution to (4.72) for the 2-point function is then:

$$\langle\langle \Phi^{\bar{m}}_{m}(z,\bar{z}) \Phi^{-\bar{m}}_{-m}(0)\rangle\rangle \propto \frac{1}{z^{2\Delta^{\bar{m}}_{j}}} \tag{4.76}$$

We now turn to the four-point function of the fields Φ. As above, the conformal blocks can be written as products of $SL(2)_k$ and $U(1)_{-k}$ factors, and eq. (4.74) for the the $SL(2)$ factor becomes

$$\begin{gathered} F_{SL(2)} = [(y_1-y_3)(y_2-y_4)]^{-2j}[(z_1-z_3)(z_2-z_4)]^{-\frac{2j(j+1)}{k+2}} G(x,t) \\ \left\{(k+2)x\frac{\partial}{\partial x} - \frac{1-t}{1-x}(x-t)\frac{\partial}{\partial t}t\frac{\partial}{\partial t} - (1+4j)t\frac{\partial}{\partial t} - 2j^2\frac{t+1}{t-1}\right\} G(x,t) = 0 \end{gathered} \tag{4.77}$$

Unlike the cases we have considered up to now, this partial differential equation is difficult to solve exactly. In [21], a solution was found for the behaviour as $t \to 1$, but to find the behaviour of four point functions of the primary fields Φ^{-m}_{m} we need to know the behaviour to all orders in t, even if we are only interested in the region where $x \to 0$ or $x \to 1$. Even so, the solution from [21] is interesting because it has a logarithmic dependence on t and x which presumably will also occur in the full solution. The solution to leading order in $t-1$ is:

$$G(x,t) \sim \left[\frac{t-1}{\log x}\right]^{-2j} \left[2\psi(1) - \psi(-2j) - \log(-k-2) - \log\left(\frac{t-1}{\log x}\right) + \cdots\right] \tag{4.78}$$

The $\log(t-1)$ term indicates the appearance in the operator product expansion of an operator $\mathcal{O}$ for which, instead of (4.75), we have the expansion

$$\mathcal{O}(z,y) = \sum_{m} \left(\mathcal{O}_m(z) + \tilde{\mathcal{O}}_m(z)\log y\right) y^{j+m} \tag{4.79}$$

It can be seen from (4.68) that $\mathcal{O}_m(z)$ and $\tilde{\mathcal{O}}_m(z)$ form a representation in which t^3 cannot be diagonalized, and has the Jordan block structure:

$$\begin{aligned} t^3\tilde{\mathcal{O}}_m(z) &= m\tilde{\mathcal{O}}_m(z) \\ t^3\mathcal{O}_m(z) &= m\mathcal{O}_m(z) + \tilde{\mathcal{O}}_m(z) \end{aligned} \tag{4.80}$$

This is similar to the situation when a four point function has a logarithmic dependence on x, which indicates the appearance in the OPE of logarithmic operators, which form Jordan blocks for the Virasoro generator L_0 [26]. This is not surprising, as it was predicted in [25] that logarithmic operators should appear in the spectrum of the $2D$ black hole. However, the operators $\mathcal{O}_m(z)$ which exist in the $SL(2,\mathcal{R})/U(1)$ coset model and in gravitationally dressed CFT, and also presumably in other coset models based on non-compact groups, differ from the logarithmic operators that have been studied up to now in that they form indecomposable representations of both the Kac-Moody and Virasoro algebras. In an ordinary logarithmic CFT, there is a pair of operators C and D which satisfy:

$$\begin{aligned} L_0 C &= \Delta C \\ L_0 D &= \Delta D + C \end{aligned} \tag{4.81}$$

In the $SL(2,\mathcal{R})$ WZNW model, the operator $\mathcal{O}(z,y)$, although it is in a reducible but indecomposable representation of $SL(2,\mathcal{R})$, is an ordinary primary field of the Virasoro algebra, since using eqs. (4.79) and (4.68) together with $J_0^A\mathcal{O}(z,y) = t^A\mathcal{O}(z,y)$ we find as usual:

$$L_0^{(WZNW)}\mathcal{O}(z,y) = \frac{1}{k+2}J_0^A J_0^A\mathcal{O}(z,y) = \frac{j(j+1)}{k+2}\mathcal{O}(z,y) \tag{4.82}$$

However, in the coset model we have:

$$L_0\mathcal{O} = \frac{1}{k+2}t_0^A t_0^A\mathcal{O} - \frac{1}{k}\bar{t}_0^3\bar{t}_0^3\mathcal{O} \tag{4.83}$$

which follows from (2.37) (the analysis leading to eq. (2.35) is the same for the operators $\mathcal{O}$ as for ordinary operators). Using eq. (4.80) we now find:

$$\begin{aligned} L_0\tilde{\mathcal{O}}_m &= \left(\frac{j(j+1)}{k+2} - \frac{\bar{m}^2}{k}\right)\tilde{\mathcal{O}}_m \\ L_0\mathcal{O}_m &= \left(\frac{j(j+1)}{k+2} - \frac{\bar{m}^2}{k}\right)\mathcal{O}_m - \frac{2\bar{m}}{k}\tilde{\mathcal{O}}_m \end{aligned} \tag{4.84}$$

If we now put $C = -\frac{2\bar{m}}{k}\tilde{\mathcal{O}}_m$ and $D = \mathcal{O}_m$, this takes the usual form (4.81). We can therefore see that operators which form indecomposable representations of $SL(2,\mathcal{R})$ are primary operators in the WZNW model but become logarithmic operators after the model is gauged.

5 Mix of gravitational and gauge dressings

One can consider a gauged WZNW model coupled to 2D gravity. As is well known 2D gravity affects both anomalous conformal dimensions and correlation functions [18]. Therefore, it is interesting to study the gravitational effects on the gauged WZNW models. Our starting point, as usual, will be the classical equation of motion of the gauged WZNW model.

In the presence of 2D gravity taken in the light cone gauge [17]

$$\bar{h} = 0, \tag{5.85}$$

the equation of motion is given as follows

$$-\frac{k}{2}\left[\partial \;+\; h\bar{\partial} \;+\; \bar{\Delta}(\bar{\partial}h)\right] g \;-\; \frac{k}{2} gA = \mathcal{J} g, \tag{5.86}$$

where $\bar{\Delta}$ is the conformal dimension of g, classically, $\bar{\Delta} = 0$. Here $\mathcal{J}$ is an affine current which obeys the same commutation relations as J in eqs. (2.10).

At the quantum level due to renormalization of the singular products, the equation of motion becomes

$$\left[\partial \;+\; h\bar{\partial} \;+\; \bar{\Delta}(\bar{\partial}h)\right] g \;-\; \eta g A \;+\; \frac{2}{\kappa}\mathcal{J} g = 0. \tag{5.87}$$

Because in the light cone gauge 2D gravity does not change the residual symmetries, the constants $\eta,\ \kappa$ will be the same as without gravity,

$$\eta = \frac{k + 2c_V(H)}{k + c_V(H)}, \qquad \kappa = \frac{1}{k + c_V(G)}. \tag{5.88}$$

Whereas the dressed conformal dimension $\bar{\Delta}$ is to be found from the gravitational Ward identities combined with the gauge Ward identities.

The gravitational Ward identities universal for all CFT's have been derived in [17]. In particular,

$$\begin{aligned} &\ll h(z)\Phi_1(z_1,\bar{z}_1)....\Phi_N(z_N,\bar{z}_N) \gg \\ &= \; -\frac{1}{\gamma}\sum_{i=1}^{N}\left[\frac{(\bar{z}-\bar{z}_i)^2}{z-z_i}\frac{\partial}{\partial \bar{z}_i} \;-\; 2\bar{\Delta}_i \frac{\bar{z}-\bar{z}_i}{z-z_i}\right] \ll \Phi_1(z_1,\bar{z}_1)....\Phi_N(z_1,\bar{z}_N) \gg . \end{aligned} \tag{5.89}$$

Here the constant γ is given as follows

$$\gamma \equiv K+2 = \frac{c-13 \pm \sqrt{(c-1)(c-25)}}{12}, \tag{5.90}$$

where

$$c = \frac{k \dim G}{k + c_V(G)} \; - \; \frac{k \dim H}{k + c_V(H)}, \tag{5.91}$$

and the +sign is chosen for $c \geq 25$, and the −sign for $c \leq 1$.

By using this identity and its derivative with respect to $\bar{z}$ in combination with the gauge Ward identities, one can finally arrive at the following equation

$$\begin{aligned} &\left\{ \frac{1}{2}\frac{\partial}{\partial z_i} \; + \; \sum_{j \neq i}^{N} \left[\frac{t_i^A t_j^A}{k + c_V(G)} \; - \; \frac{\bar{t}_i^a \bar{t}_j^a}{k + c_V(H)} \right] \frac{1}{z_i - z_j} \; + \; \frac{1}{2\gamma} \sum_{j \neq i}^{N} \left[\frac{(\bar{z}_i - \bar{z}_j)^2}{z_i - z_j} \frac{\partial^2}{\partial \bar{z}_i \partial \bar{z}_j} \right.\right. \\ &\left.\left. + \; 2\bar{\Delta} \frac{\bar{z}_i - \bar{z}_j}{z_i - z_j} \left(\frac{\partial}{\partial \bar{z}_j} \; - \; \frac{\partial}{\partial \bar{z}_i} \right) \; - \; \frac{2\bar{\Delta}^2}{z_i - z_j} \right] \right\} \ll \Phi(z_1, \bar{z}_1) .... \Phi(z_N, \bar{z}_N) \gg = 0. \end{aligned} \tag{5.92}$$

The latter equation can be presented as follows

$$\left\{ \frac{1}{2}\frac{\partial}{\partial z_i} \; + \; \sum_{j \neq i}^{N} \frac{\frac{t_i^A t_j^A}{k + c_V(G)} \; - \; \frac{\bar{t}_i^a \bar{t}_j^a}{k + c_V(H)} \; - \; \frac{\eta_{\alpha\beta} \bar{T}_i^\alpha \bar{T}_j^\beta}{K+2}}{z_i - z_j} \right\} \ll \Phi_1(z_1, \bar{z}_1) .... \Phi_N(z_N, \bar{z}_N) \gg = 0, \tag{5.93}$$

where the $SL(2)$ generators $\bar{T}_i^\alpha$ are given as follows

$$\bar{T}_i^\alpha = \bar{z}_i^{\alpha+1} \frac{\partial}{\partial \bar{z}_i} \; - \; (\alpha + 1)\, \bar{\Delta}_i \, \bar{z}_i^\alpha, \qquad \alpha = -1, \; 0, \; +1. \tag{5.94}$$

In the case of the two-point function, one can obtain the following relation for the dressed conformal dimensions:

$$\Delta_i \; - \; \Delta_{0i} = \frac{\bar{\Delta}_i(\bar{\Delta}_i - 1)}{\gamma}, \tag{5.95}$$

where Δ_{0i} is the conformal dimension without gravity. ¿From our differential equation itself, it does not follow that the holomorphic Δ_i and the antiholomorphic $\bar{\Delta}_i$ conformal dimensions coincide. However, Lorentz symmetry imposes one extra condition on the conformal dimensions, which is as follows

$$\Delta_i \; - \; \Delta_{0i} = \bar{\Delta}_i \; - \; \bar{\Delta}_{0i}. \tag{5.96}$$

This constraint guarantees that the Lorentz spin of operators is not affected by the gravitational dressing. Thus, our differential equation gives rise to the known KPZ formula

$$\bar{\Delta}_i \; - \; \bar{\Delta}_{0i} = \frac{\bar{\Delta}_i(\bar{\Delta}_i - 1)}{\gamma}, \tag{5.97}$$

6 Conclusion

By using the Polyakov chiral gauge approach, we have derived a few KZ-type equations which allow us to compute correlation functions of gauged WZNW models. We have checked that our equations give the correct expressions for correlators of the minimal models. In the case of Witten's 2D black hole, we saw that our equation closely resembles the equation describing the gravitational dressing. It might be interesting to pursue further this analogy to consider the case of the continuous unitary representations of $SL(2)$. As is known these representations admit complex values for the spin j [29]. Since, following the above analogy with 2D gravity, j is to be identified with the dressed conformal dimension, it is curious to know what implications would this fact mean for 2D gravity?

Another interesting question is about a relation between our equation (2.35) and the equation obtained in [12] on the bases of the affine-Virasoro construction. At least for the minimal models both the equations give rise to the correct correlation functions. At the same time, it is obvious that the two equations have different structures. Indeed, the equation in [12] has its connection W_i already expressed in terms of some correlation functions. We can think of one possible explanation to the existence of two different equations having one and the same solutions. Namely, this situation may occur in gauge invariant theories when one uses two different gauges. However, we do not know what gauge could lead to the equation in [12], because it is not clear to us how this equation can be derived directly from the equations of motion of the gauged WZNW model. We also would like to point out other generalizations of the KZ equation considered in [30]

The equations which we have studied in the present paper have many more solutions than we have actually discussed. Some of them may be important for a better understanding of the dynamical properties of the gauged WZNW models. Especially, it is interesting

to look for solutions to eq. (4.74) corresponding to the continuous representations as they appear to be very significant for 2D black holes.

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